

- 3 The identity sign $\equiv$ is sometimes used but normally we use an ordinary '=' sign when proving identities. Remember that an identity is true for *all* values of the variable.

For example $\tan \theta \equiv \frac{\sin \theta}{\cos \theta}$ for all values of θ .

Example 5

Prove the identity $\frac{\cos^2 \theta}{1 - \sin \theta} = 1 + \sin \theta$.

$$\begin{aligned} \text{L.H.S.} &= \frac{\cos^2 \theta}{1 - \sin \theta} = \frac{1 - \sin^2 \theta}{1 - \sin \theta} \\ &= \frac{(1 - \sin \theta)(1 + \sin \theta)}{(1 - \sin \theta)} \quad [\text{Difference of two squares.}] \\ &= 1 + \sin \theta \text{ as required.} \end{aligned}$$

Example 6

Prove the identity $\frac{\cos \theta}{1 + \cos \theta} = \cot \theta (\operatorname{cosec} \theta - \cot \theta)$.

In this example we could start with either side.

$$\begin{aligned} \text{L.H.S.} &= \frac{\cos \theta}{1 + \cos \theta} = \frac{\cos \theta(1 - \cos \theta)}{(1 + \cos \theta)(1 - \cos \theta)} \quad [\text{Multiply numerator and denominator by } (1 - \cos \theta).] \\ &= \frac{\cos \theta(1 - \cos \theta)}{1 - \cos^2 \theta} \\ &= \frac{\cos \theta(1 - \cos \theta)}{\sin^2 \theta} \\ &= \frac{\cos \theta}{\sin \theta} \left(\frac{1 - \cos \theta}{\sin \theta} \right) \\ &= \cot \theta (\operatorname{cosec} \theta - \cot \theta) \text{ as required.} \end{aligned}$$

[See for yourself that you can prove this identity just as well by starting with the right hand side.]

EXERCISE 1

- 1 Evaluate the following. Give answers correct to 3 dp where necessary.

a $\cot 30^\circ$

b $\operatorname{cosec} 60^\circ$

c $\sec 10^\circ$

d $\operatorname{cosec} 45^\circ$

e $\cot 72^\circ$

f $\sec^2 30^\circ$

- 2 Simplify the expressions.

a $1 + \tan^2 x$

b $1 + \tan^2 2x$

c $1 - \cos^2 x$

d $\operatorname{cosec} x \tan x$

e $\cot \theta(1 - \cos^2 \theta)$

f $1 + \cot^2 \theta$

3 Prove the following identities.

$$\begin{array}{ll} \text{a } (\sin \theta + \cos \theta)^2 - 2 \sin \theta \cos \theta = 1 & \text{b } \tan \theta \operatorname{cosec} \theta = \sec \theta \\ \text{c } \operatorname{cosec} \theta (1 - \cos \theta)(1 + \cos \theta) = \sin \theta & \text{d } \cot \theta \sec \theta \tan \theta \sqrt{1 - \sin^2 \theta} = 1 \\ \text{e } \frac{1}{\sec^2 \theta} + \frac{1}{\operatorname{cosec}^2 \theta} = 1 & \text{f } \sec \theta - \cos \theta = \sin \theta \tan \theta \\ \text{g } \sin^2 \theta + 2 \cos^2 \theta = 2 - \sin^2 \theta & \text{h } \tan \theta + \cot \theta = \frac{1}{\sin \theta \cos \theta} \end{array}$$

4 Solve the equations for $0 \leq x \leq 360^\circ$.

$$\begin{array}{lll} \text{a } \sec x = 2 & \text{b } \cot x = \sqrt{3} & \text{c } \operatorname{cosec} x = \sqrt{2} \\ \text{d } \sec x = 1.2 & \text{e } \cot x = 3 & \text{f } \operatorname{cosec} x = 1 \end{array}$$

5 Prove the identities.

$$\begin{array}{ll} \text{a } \operatorname{cosec} \theta - \sin \theta = \cot \theta \cos \theta & \text{b } \cos^2 \theta - \sin^2 \theta = 2 \cos^2 \theta - 1 \\ \text{c } \operatorname{cosec}^2 \theta + \sec^2 \theta = \operatorname{cosec}^2 \theta \sec^2 \theta & \text{d } \frac{\sin^2 \theta}{1 - \cos \theta} = 1 + \cos \theta \\ \text{e } \frac{\sin \theta}{1 + \sin \theta} = \tan \theta (\sec \theta - \tan \theta) & \text{f } \operatorname{cosec}^2 \theta (\tan^2 \theta - \sin^2 \theta) = \tan^2 \theta \\ \text{g } \frac{\sec \theta}{\tan \theta + \cot \theta} = \sin \theta & \text{h } \frac{\cos \theta}{\sin \theta + 1} + \frac{\sin \theta + 1}{\cos \theta} = 2 \sec \theta \end{array}$$

6 Sketch the curve $y = \sec x$ for $0 \leq x < 90^\circ$.

7 Sketch the curve $y = \operatorname{cosec} x$ for $0 \leq x \leq 180^\circ$.

8 Solve the equations.

$$\begin{array}{l} \text{a } 2 \cot^2 \theta - 3 \cot \theta + 1 = 0 \text{ for } 0 \leq \theta \leq 360^\circ \\ \text{b } \sec^2 \theta - \tan \theta = 1 \text{ for } -180^\circ \leq \theta \leq 180^\circ \\ \text{c } \cot^2 \theta - 3 \operatorname{cosec} \theta + 3 = 0 \quad 0 \leq \theta \leq 180^\circ \end{array}$$

9 Prove the identities.

$$\begin{array}{ll} \text{a } \sin^4 x - \cos^4 x = \sin^2 x - \cos^2 x & \text{b } \sec^4 x - \tan^4 x = 1 + 2 \tan^2 x \\ \text{c } \frac{\sin \theta}{1 - \cos \theta} + \frac{\sin \theta}{1 + \cos \theta} = 2 \operatorname{cosec} \theta & \text{d } \frac{\cot^2 \theta}{1 + \cot^2 \theta} = \cos^2 \theta \\ \text{e } \frac{1 - \tan^2 \theta}{1 + \tan^2 \theta} = 1 - 2 \sin^2 \theta & \text{f } \frac{2 \tan \theta}{1 + \tan^2 \theta} = 2 \sin \theta \cos \theta \\ \text{g } \frac{\cot \theta + \tan \theta}{\operatorname{cosec} \theta + \sec \theta} = \frac{1}{\cos \theta + \sin \theta} & \\ \text{h } \sin^3 x + \cos^3 x = (\sin x + \cos x)(1 - \sin x \cos x) & \end{array}$$

- 10 Solve the inequality $\operatorname{cosec} \theta < 2$ for values of θ between 0 and 180° .
- 11 a Simplify the expression $(1 - \sin \theta)(1 + \sin \theta) \sec \theta$.
 b Solve the inequality $(1 - \sin \theta)(1 + \sin \theta) \sec \theta > \frac{1}{2}$ for values of θ between -90° and 90° .
- 12 Solve the equations
 a $\sec 2x = 3$ for $0 \leq x \leq 360^\circ$
 b $3 \operatorname{cosec}^2 2x = 4$ for $0 \leq x \leq 180^\circ$
- 13 Solve the simultaneous equations

$$\operatorname{cosec} 2x = \sqrt{2}$$

$$\cot(x + y) = \sqrt{3}$$
 for values of x and y from 0° to 180° .

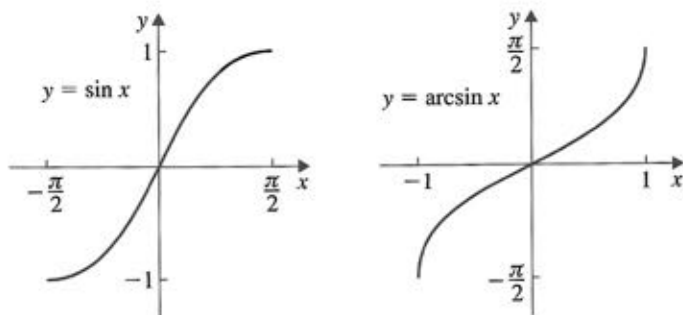
2.2 Inverse trigonometric functions (arcsin x , arccos x and arctan x)

We know from the work with functions that only one-to-one functions have inverses. We also know that $\sin$, $\cos$ and $\tan$ are all many-to-one functions unless we restrict the domain.

By considering the graph of $y = \sin x$ we can see that it is one-to-one if we restrict the domain to $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$.

If we restrict the domain in this way then there is an inverse, called arcsin. We write $x = \arcsin y$ or $x = \sin^{-1} y$.

The graph of $y = \arcsin x$ is obtained by reflecting $y = \sin x$ about the line $y = x$. We have the following graphs.



The domain of $y = \arcsin x$ is $-1 \leq x \leq 1$ and its range is $-\frac{\pi}{2} \leq \arcsin x \leq \frac{\pi}{2}$.

By considering the graph of $y = \cos x$ we can see that it is one-to-one if we restrict the domain to $0 \leq x \leq \pi$.

If we restrict the domain in this way then there is an inverse, called arccos. We write $x = \arccos y$ or $x = \cos^{-1} y$.

The graph of $y = \arccos x$ is obtained by reflecting $y = \cos x$ about the line $y = x$. We have the following graphs.

b $\arctan(-\sqrt{3})$ means 'the angle whose tan is $-\sqrt{3}$ '

We know that $\tan\left(-\frac{\pi}{3}\right) = -\sqrt{3}$.

$$\therefore \arctan(-\sqrt{3}) = -\frac{\pi}{3}$$

EXERCISE 2

In Questions 1 and 2 do not use a calculator.

1 Evaluate the following in degrees.

a $\arcsin \frac{1}{2}$

b $\arccos 0$

c $\arctan 1$

d $\arccos(-\frac{1}{2})$

e $\arctan\left(-\frac{1}{\sqrt{3}}\right)$

f $\arcsin\left(\frac{\sqrt{3}}{2}\right)$

2 Evaluate the following in radians in terms of π .

a $\arctan(-1)$

b $\arccos(\frac{1}{2})$

c $\arcsin\left(\frac{1}{\sqrt{2}}\right)$

d $\arccos(1)$

e $\arctan(\sqrt{3})$

f $\arcsin(-1)$

3 Sketch the graph of $f(x) = \arcsin x$ for $-1 \leq x \leq 1$.

4 Sketch the graph of $f(x) = \arccos x$ for $-1 \leq x \leq 1$.

5 Solve the equations, for $0 \leq x \leq 360^\circ$.

a $\sin x = \cos(\arcsin \frac{1}{2})$

b $\tan x = 2 \sin\left(\arccos \frac{\sqrt{3}}{2}\right)$

c $\cos x = \sin\left(\arctan \frac{1}{\sqrt{3}} + \arccos \frac{1}{2}\right)$

6 Simplify the following.

a $\sin(\arcsin \frac{1}{2})$

b $\tan\left(\arccos \frac{1}{\sqrt{2}}\right) - \sin(\arccos 0)$

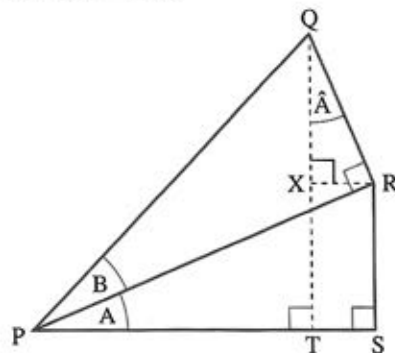
c $\cos(\arccos \theta)$

d $3\theta - 2 \tan(\arctan \theta)$

2.3 Compound angles; sin, cos and tan of $(A \pm B)$

In the diagram $\hat{A} = \hat{PRX}$
 $= 90^\circ - \hat{XRQ}$
 $= \hat{RQX}$

$$\begin{aligned} \sin(A+B) &= \frac{QT}{PQ} = \frac{TX + XQ}{PQ} \\ &= \frac{TX}{PQ} + \frac{XQ}{PQ} = \frac{RS}{PQ} + \frac{XQ}{PQ} \\ &= \frac{RS}{PR} \times \frac{PR}{PQ} + \frac{XQ}{QR} \times \frac{QR}{PQ} \end{aligned}$$



EXERCISE 3

- 1 a Write down the formula for $\sin(A + B)$.
b Evaluate $\sin 80^\circ \cos 10^\circ + \cos 80^\circ \sin 10^\circ$
- 2 a Write down the formula for $\cos(A + B)$.
b Evaluate $\cos 50^\circ \cos 40^\circ - \sin 50^\circ \sin 40^\circ$
- 3 Answer 'true' or 'false':
a $\tan(A + 45^\circ) = \tan A + 1$ b $\tan(A + 45^\circ) = \frac{\tan A + 1}{1 - \tan A}$
- 4 Simplify the following
a $\sin \theta \cos 2\theta + \cos \theta \sin 2\theta$ b $\cos 3A \cos A + \sin 3A \sin A$
c $\sin 5x \cos 2x - \cos 5x \sin 2x$ d $\cos B \cos B + \sin B \sin B$
- 5 Find the exact values of the following (using surds):
a $\sin(60^\circ + 45^\circ)$ b $\sin 105^\circ$ c $\cos 75^\circ$
d $\cos 15^\circ$ e $\tan(-15^\circ)$
- 6 Given that A and B are acute angles such that $\sin A = \frac{5}{13}$ and $\cos B = \frac{3}{5}$, find the exact values of:
a $\sin(A + B)$ b $\cos(A - B)$ c $\tan(A + B)$
- 7 Solve the equation $\sin(x + 60^\circ) + \cos(x + 30^\circ) = \frac{1}{2}$ in the range $0^\circ \leq x < 360^\circ$.
- 8 Express the following as single trigonometric expressions:
a $\cos(x + 90^\circ)$ b $\sin(x - 90^\circ)$ c $\sin(x + 180^\circ)$ d $\cos(x - 180^\circ)$
- 9 Given that $\tan(x + 45^\circ) = 2$, find the value of $\tan x$, without using a calculator.
- 10 Given that $\tan(A + B) = 3$ and that $\tan B = \frac{1}{2}$, find the exact value of $\tan A$.
- 11 Solve the equation $\sin(\theta + 45^\circ) = \sqrt{2} \cos \theta$, for $0^\circ \leq \theta \leq 360^\circ$.
- 12 Find the greatest value that the following expressions can have and state the value of θ between 0° and 360° , for which these values occur.
a $\sin \theta \cos 40^\circ + \cos \theta \sin 40^\circ$ b $\sin \theta \cos 15^\circ - \cos \theta \sin 15^\circ$
- 13 Prove the following identities:
a $\sin(\theta + 90^\circ) \equiv \cos \theta$ b $\sin(A + B) + \sin(A - B) \equiv 2 \sin A \cos B$
c $\cos x \cos 2x + \sin x \sin 2x \equiv \cos x$ d $\sin 2x \cos x - \cos 2x \sin x \equiv \sin x$
e $\frac{\sin(A + B)}{\sin(A - B)} \equiv \frac{\tan A + \tan B}{\tan A - \tan B}$ f $\tan A + \tan B \equiv \frac{\sin(A + B)}{\cos A \cos B}$
g $\sin(A + B) \sin(A - B) \equiv \sin^2 A - \sin^2 B$

- 14 Using $\tan(x + \alpha)$ for a suitable value of α , express $\frac{1 + \tan x}{1 - \tan x}$ as a single trigonometric expression.
- 15 a Find an expression for $\tan 2A$ in terms of $\tan A$.
 b Given that $\tan \theta = \frac{4}{3}$ and that θ is acute find the exact value of $\tan\left(\frac{\theta}{2}\right)$.
- 16 If $\sin(x - A) = \cos(x + A)$, show that $\tan x = 1$.
- 17 Find the exact values of the following:
 a $\cot 15^\circ$ (in the form $a + \sqrt{b}$)
 b $\sec 75^\circ$ (in the form $\sqrt{a} + \sqrt{b}$)
 c $\operatorname{cosec} 105^\circ$ (in the form $\sqrt{a} - \sqrt{b}$)
- 18 Show that $\operatorname{cosec}(A + B) \equiv \frac{\sec A \sec B}{\tan A + \tan B}$.
- 19 Show that $\cot(A + B) \equiv \frac{\cot A \cot B - 1}{\cot A + \cot B}$.
- 20 Prove that if the sum of A , B and C is 180° then $\frac{\tan A + \tan B + \tan C}{\tan A \tan B \tan C} \equiv 1$.
- 21 Show that $\tan 15^\circ = 2 - \sqrt{3}$.
- 22 Given $\sin(A + B) = k \sin(A - B)$, show that $\tan A = \frac{k + 1}{k - 1} \tan B$.
- 23 Given that $\tan(A + B) = 1$ and that $\tan A = \frac{1}{2}$, find $\tan B$.
- 24 a Given that $\sin(x + \alpha) = 2 \sin(x - \alpha)$ show that $\tan x = 3 \tan \alpha$.
 b Use the result of part a to solve the equation $\sin\left(2y + \frac{\pi}{4}\right) = 2 \sin\left(2y - \frac{\pi}{4}\right)$ for $0 < y < \pi$.

2.4 The double angle formulae

- We know that $\sin(A + B) = \sin A \cos B + \cos A \sin B$

Put $B = A$ in this identity.

$$\sin(A + A) = \sin A \cos A + \cos A \sin A$$

$$\text{or } \sin 2A = 2 \sin A \cos A \quad [1]$$

- In a similar way we can prove that

$$\cos 2A = \cos^2 A - \sin^2 A \quad [2]$$

Using $\sin^2 A + \cos^2 A = 1$, we can show further that

$$\cos 2A = 2 \cos^2 A - 1 \quad [2a]$$

$$\cos 2A = 1 - 2 \sin^2 A \quad [2b]$$

- Finally $\tan 2A = \frac{2 \tan A}{1 - \tan^2 A} \quad [3]$

Factorising, $\cos \theta(2 \sin \theta - 1) = 0$

Either $\cos \theta = 0 \Rightarrow \theta = 90^\circ$ or 270°

Or $2 \sin \theta - 1 = 0$
 $\sin \theta = \frac{1}{2} \Rightarrow \theta = 30^\circ$ or 150°

So finally $\theta = 30^\circ, 90^\circ, 150^\circ, 270^\circ$

Notice that in line [1] we did not divide through by $\cos \theta$. If we did that we would lose all the solutions we obtain from $\cos \theta = 0$. The correct method is to factorise as above.

Example 4

Prove the identity $\frac{\cos 2A}{\cos A + \sin A} = \cos A - \sin A$

Start with the more complicated side and rearrange it until we obtain the expression on the other side. In this case the more complicated side is the left hand side (L.H.S.).

$$\begin{aligned}\text{L.H.S.} &= \frac{\cos 2A}{\cos A + \sin A} \\&= \frac{\cos^2 A - \sin^2 A}{\cos A + \sin A} \\&= \frac{(\cos A - \sin A)(\cos A + \sin A)}{\cos A + \sin A} \quad [\text{Difference of two squares.}] \\&= \cos A - \sin A, \text{ as required.}\end{aligned}$$

Example 5

Prove the identity $\cos 3A = 4 \cos^3 A - 3 \cos A$

$$\begin{aligned}\text{Write } \cos 3A &= \cos(2A + A) \\&= \cos 2A \cos A - \sin 2A \sin A \\&= (2 \cos^2 A - 1) \cos A - (2 \sin A \cos A) \sin A \\&= 2 \cos^3 A - \cos A - 2 \sin^2 A \cos A \\&= 2 \cos^3 A - \cos A - 2(1 - \cos^2 A) \cos A \\&= 2 \cos^3 A - \cos A - 2 \cos A + 2 \cos^3 A \\&= 4 \cos^3 A - 3 \cos A \text{ as required.}\end{aligned}$$

EXERCISE 4

1 Evaluate the following, without using a calculator.

a $2 \sin 45^\circ \cos 45^\circ$

b $2 \cos^2 15^\circ - 1$

c $1 - 2 \sin^2 30^\circ$

d $\frac{2 \tan 22\frac{1}{2}^\circ}{1 - \tan^2 22\frac{1}{2}^\circ}$

2 Express in a more simple form.

$$\begin{array}{llll} \text{a } 2 \sin 10^\circ \cos 10^\circ & \text{b } 2 \cos^2 17^\circ - 1 & \text{c } 1 - 2 \sin^2 35^\circ & \text{d } \frac{2 \tan 11^\circ}{1 - \tan^2 11^\circ} \\ \text{e } \sin \theta \cos \theta & \text{f } \frac{\tan 2\theta}{1 - \tan^2 2\theta} & \text{g } \frac{\sin 2A}{\cos A} & \text{h } 2 \sin^2 A + \cos 2A \end{array}$$

3 Find the values of $\sin 2\theta$ and $\cos 2\theta$, given that θ is acute and that $\sin \theta = \frac{3}{5}$.

4 Find the values of $\sin 2\theta$ and $\cos 2\theta$, given that θ is acute and that $\cos \theta = \frac{5}{13}$.

5 Find $\tan 2\theta$ given θ is acute and,

$$\text{a } \tan \theta = \frac{1}{2} \quad \text{b } \sin \theta = \frac{3}{5}$$

6 Find $\sin x$, if $\cos 2x = \frac{1}{8}$

7 Find $\cos x$, if $\cos 2x = \frac{7}{25}$

8 Find the possible values of $\tan \frac{1}{2}\theta$, if $\tan \theta = \frac{3}{4}$

In Questions 9 to 18 solve the equations, for $0 \leq \theta \leq 360^\circ$.

$$9 \sin 2\theta - \sin \theta = 0$$

$$10 2 \sin 2\theta + \cos \theta = 0$$

$$11 4 \cos 2\theta - 6 \cos \theta + 5 = 0$$

$$12 \sin 2\theta = 2 \cos 2\theta$$

$$13 \cos 2\theta + \sin \theta - 1 = 0$$

$$14 3 \cos 2\theta - \cos \theta + 2 = 0$$

$$15 \tan 2\theta + \tan \theta = 0$$

$$16 \sin \theta + \sin \frac{\theta}{2} = 0$$

$$17 \sin \frac{\theta}{2} = \sin \theta$$

$$18 2 \cos 2x = 5 - 13 \sin x$$

19 Prove the identity $2 \operatorname{cosec} 2\theta = \sec \theta \operatorname{cosec} \theta$.

$$\left[\text{Hint: Start with 'L.H.S.} = \frac{2}{\sin 2\theta} \right]$$

20 Prove the identity $\tan 2\theta \sec \theta = 2 \sin \theta \sec 2\theta$.

$$\left[\text{Hint: start with 'L.H.S.} = \frac{\sin 2\theta}{\cos 2\theta} \times \frac{1}{\cos \theta} \right]$$

In Questions 21 to 35 prove the identities.

$$21 \frac{\cos 2A}{\cos A - \sin A} = \cos A + \sin A$$

$$22 \frac{\sin A}{\sin B} + \frac{\cos A}{\cos B} = \frac{2 \sin(A+B)}{\sin 2B}$$

$$23 \frac{\sin A}{\sin B} - \frac{\cos A}{\cos B} = \frac{2 \sin(A-B)}{\sin 2B}$$

$$24 \tan A + \cot A = 2 \operatorname{cosec} 2A$$

$$25 \cot A - \tan A = 2 \cot 2A$$

$$26 \frac{\sin 2A}{1 + \cos 2A} = \tan A$$

$$27 \quad \sin 2A = \frac{2 \tan A}{1 + \tan^2 A}$$

$$28 \quad \cos 2A = \frac{1 - \tan^2 A}{1 + \tan^2 A}$$

$$29 \quad \sin 3A = 3 \sin A - 4 \sin^3 A$$

$$30 \quad \cos 3A = 4 \cos^3 A - 3 \cos A$$

$$31 \quad \cot \theta = \frac{\sin 2\theta}{1 - \cos 2\theta}$$

$$32 \quad \cos^4 \theta - \sin^4 \theta = \cos 2\theta$$

$$33 \quad \cot 2A + \operatorname{cosec} 2A = \cot A$$

$$34 \quad \sin^4 \theta + \cos^4 \theta = \frac{1}{4}(\cos 4\theta + 3)$$

$$35 \quad \tan 3\theta = \frac{3 \tan \theta - \tan^3 \theta}{1 - 3 \tan^2 \theta}$$

$$36 \quad \text{Use the double angle formula for cosine to show that } \cos x = 2 \cos^2\left(\frac{x}{2}\right) - 1.$$

$$37 \quad \text{Express } \sin x \text{ in terms of } \sin\left(\frac{x}{2}\right) \text{ and } \cos\left(\frac{x}{2}\right).$$

$$38 \quad \text{Express } \tan x \text{ in terms of } \tan\left(\frac{x}{2}\right).$$

$$39 \quad \text{a Show that } \sin^2\left(\frac{x}{2}\right) = \frac{1}{2}(1 - \cos x).$$

$$\text{b Show that } \cos^2\left(\frac{x}{2}\right) = \frac{1}{2}(1 + \cos x).$$

$$40 \quad \text{Prove that } \tan 22\frac{1}{2}^\circ = \sqrt{2} - 1.$$

$$41 \quad \text{a Show that } \operatorname{cosec} 2x - \cot 2x = \tan x$$

$$\text{b Hence find the exact value of } \tan 75^\circ, \text{ giving your answer in the form } a + b\sqrt{3}.$$

$$42 \quad \text{Solve the equation } 4 \tan 2\theta = \cot \theta, \text{ for } 0^\circ < \theta < 360^\circ, \text{ giving your answers correct to 1 decimal place.}$$

$$43 \quad \text{Express } 8 \cos^4 \theta \text{ in the form } A \cos 4\theta + B \cos 2\theta + C, \text{ giving the numerical values of the constants } A, B \text{ and } C.$$

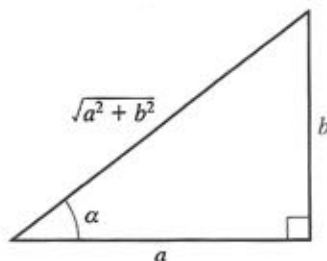
2.5 The form $a \cos \theta + b \sin \theta$

A special technique is used to express $a \cos \theta + b \sin \theta$ in the form $R \cos(\theta - \alpha)$, where α is an acute angle.

$$\text{We write } a \cos \theta + b \sin \theta = \sqrt{a^2 + b^2} \left[\frac{a}{\sqrt{a^2 + b^2}} \cos \theta + \frac{b}{\sqrt{a^2 + b^2}} \sin \theta \right]$$

$$\text{Let } \cos \alpha = \frac{a}{\sqrt{a^2 + b^2}}$$

$$\sin \alpha = \frac{b}{\sqrt{a^2 + b^2}}$$



$$\frac{4}{\sqrt{17}} \sin \theta - \frac{1}{\sqrt{17}} \cos \theta = \frac{2}{\sqrt{17}}$$

Let $\cos \alpha = \frac{4}{\sqrt{17}}$, $\sin \alpha = \frac{1}{\sqrt{17}}$. $\alpha = 14.04^\circ$ (2 decimal places)

We have $\sin \theta \cos \alpha - \cos \theta \sin \alpha = \frac{2}{\sqrt{17}}$

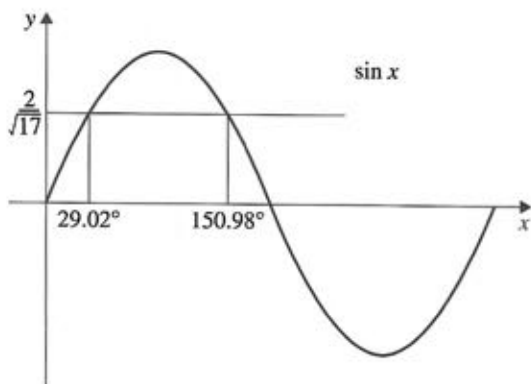
$$\sin(\theta - 14.04^\circ) = \frac{2}{\sqrt{17}}$$

$$\theta - 14.04^\circ = 29.02^\circ, 150.98^\circ$$

$$\theta = 43.1^\circ \text{ or } 165.0^\circ$$

Notice that we chose to work with α correct to 2 decimal places in the working and then gave the final answer correct to 1 decimal place.

You can of course retain the value of α in the memory of your calculator.



Example 4

Solve the equation $3 \cos \theta + 2 \sin \theta = 1$, for $0 \leq \theta \leq 2\pi$, giving your answers in radians correct to 3 decimal places.

Divide both sides by $\sqrt{3^2 + 2^2}$.

$$\frac{3}{\sqrt{13}} \cos \theta + \frac{2}{\sqrt{13}} \sin \theta = \frac{1}{\sqrt{13}}$$

Let $\cos \alpha = \frac{3}{\sqrt{13}}$, $\sin \alpha = \frac{2}{\sqrt{13}}$, so $\alpha = \cos^{-1} \frac{3}{\sqrt{13}} = 0.588 00$ (to 5 d.p.)

We have $\cos \theta \cos \alpha + \sin \theta \sin \alpha = \frac{1}{\sqrt{13}}$

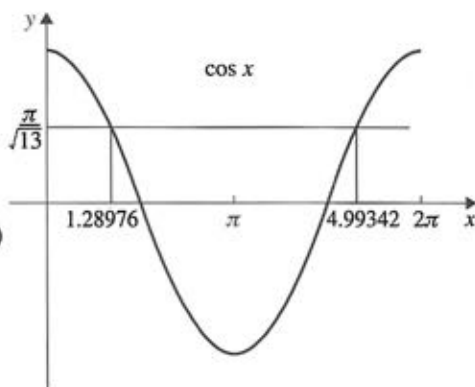
$$\cos(\theta - 0.588 00) = \frac{1}{\sqrt{13}}$$

From a calculator $\cos^{-1} \frac{1}{\sqrt{13}} = 1.289 76$ radians (correct to 5 d.p.)

$$\theta - 0.588 00 = 1.289 76 \text{ or } (2\pi - 1.289 76)$$

$$\theta - 0.588 00 = 1.289 76, 4.993 42$$

$\theta = 1.878$ rad or 5.581 rad (correct to 3 decimal places).



EXERCISE 5

1 Write each expression in the form $R \cos(\theta - \alpha)$, where $0 < \alpha < 90^\circ$.

a $4 \cos \theta + 3 \sin \theta$

b $5 \cos \theta + 12 \sin \theta$

c $\cos \theta + 2 \sin \theta$

- 2 Find the value of R and α in each of the following.
- a $3 \sin \theta + 4 \cos \theta = R \sin(\theta + \alpha)$ b $4 \cos \theta - 3 \sin \theta = R \cos(\theta + \alpha)$
 c $2 \cos \theta + \sin \theta = R \cos(\theta - \alpha)$ d $8 \sin \theta - 15 \cos \theta = R \sin(\theta - \alpha)$
 e $8 \sin \theta + 6 \cos \theta = R \sin(\theta + \alpha)$ f $\cos \theta + \sin \theta = R \cos(\theta - \alpha)$
- 3 a Express $f(\theta) = \sin \theta + \sqrt{3} \cos \theta$ in the form $R \sin(\theta + \alpha)$.
 b Find the maximum value of $f(\theta)$ and state the smallest positive value of θ that gives this maximum value.
- 4 Find the maximum value of each expression and state the smallest positive value of θ that gives this value.
- a $3 \cos \theta + 4 \sin \theta$ b $5 \sin \theta - 12 \cos \theta$
 c $\cos \theta + \sin \theta$ d $8 \cos \theta - 15 \sin \theta$
- 5 a Express $4 \cos \theta + 3 \sin \theta$ in the form $R \cos(\theta - \alpha)$, where $R > 0$ and $0 < \alpha < 90^\circ$.
 b Solve the equation $4 \cos \theta + 3 \sin \theta = 2$, for $0 \leq \theta \leq 360^\circ$.
- 6 a Express $\sin \theta + \sqrt{8} \cos \theta$ in the form $R \sin(\theta + \alpha)$.
 b Solve the equation $\sin \theta + \sqrt{8} \cos \theta = 2$, for $0 \leq \theta \leq 360^\circ$.
- 7 Solve the following equations, correct to one decimal place, for $0 \leq \theta \leq 360^\circ$.
- a $8 \cos \theta + 15 \sin \theta = 8.5$ b $5 \sin \theta + 12 \cos \theta = 7$
 c $\sin \theta + \sqrt{3} \cos \theta = 1$ d $4 \sin \theta - 3 \cos \theta = 2$
 e $4 \cos \theta - 6 \sin \theta = 5$ f $5 \sin \theta - 3 \cos \theta = 1$
 g $8 \sin \theta - 24 \cos \theta = 3$ h $5 \cos 2x + 2 \sin 2x = 3$
- 8 Solve the equations for $0 < \theta < 2\pi$, giving your answers in radians correct to two decimal places.
- a $\cos(\theta - 1.2^\circ) = \frac{1}{2}$ b $\sin(\theta + 0.4^\circ) = 0.7$
- 9 a Express $2 \cos x + \sqrt{3} \sin x$ in the form $R \cos(x - \alpha)$, where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$.
 b Find the smallest positive root of the equation $2 \cos x + \sqrt{3} \sin x = 2$, giving your answer in radians correct to two decimal places.
- 10 Solve the equations for $0 \leq \theta \leq 2\pi$, giving your answers in radians correct to two decimal places.
- a $3 \sin \theta + \cos \theta = 2$ b $\cos \theta - 2 \sin \theta = 1$
 c $\cos \theta - \sin \theta = 1$ d $3 \sin \theta - 4 \cos \theta = 3.5$
- 11 a Express $\sin \theta - 3 \cos \theta$ in the form $R \sin(\theta - \alpha)^\circ$, giving the values of R and α .
 b Explain why the equation $\sin \theta - 3 \cos \theta = 4$ has no solutions.
- 12 Find the range of values of the constant k for which the equation $\sin \theta + \sqrt{2} \cos \theta = k$, has real solutions for θ .

- 13 You are given $f(\theta) = (\cos \theta - \sin \theta)(17 \cos \theta - 7 \sin \theta)$
- Show that $f(\theta)$ may be written in the form $5 \cos 2\theta - 12 \sin 2\theta + c$, where c is a constant. State the value of c .
 - Write $5 \cos 2\theta - 12 \sin 2\theta$ in the form $R \cos(2\theta + \alpha)$, where $0 < \alpha < \frac{\pi}{2}$. State the value of R and find α in radians correct to 3 decimal places.
- 14 a Express $3 \cos \theta - 4 \sin \theta$ in the form $R \cos(\theta + \alpha)$.
- b Find the greatest possible value of $\frac{2}{(3 \cos \theta - 4 \sin \theta + 6)}$.
- 15 a Express $f(\theta) = 5 \cos \theta + 12 \sin \theta$ in the form $R \cos(\theta - \alpha)$.
- b Find the smallest possible value of $\frac{30}{f(\theta) + 2}$.
- 16 a Write $8 \sin \theta \cos \theta - 6 \sin^2 \theta$ in the form $R \sin(2\theta + \alpha) + k$.
- b Hence solve the equation $8 \sin \theta \cos \theta - 6 \sin^2 \theta = 1$, for $0 < \theta < 180^\circ$.
- 17 a Express $4 \cos x + 3 \sin x$ in the form $R \cos(x - \alpha)$.
- b Find the greatest and least values of $\frac{1}{(4 \cos x + 3 \sin x + 7)}$.
- c Find, in radians correct to 3 decimal places, the solution of the equation $(4 \cos x + 3 \sin x)^2 = 12.5$, for $0 \leq x \leq \pi$.

REVIEW EXERCISE 2

- Solve the following trigonometric equations in the given intervals (to 1 dp where necessary).

a $\sin x = \frac{2}{3}$, for $0 \leq x \leq 360^\circ$	b $\cos x = -\frac{3}{4}$, for $0 \leq x \leq 360^\circ$
c $\tan x = -0.7$, for $0 \leq x \leq 360^\circ$	d $\sin x = -\frac{1}{2}$, for $0 \leq x \leq 360^\circ$
e $\cos x = \frac{1}{2}$, for $0 \leq x \leq 2\pi$	f $\tan x = 1$, for $-\pi \leq x \leq \pi$
g $2 \sin^2 x = 3 \cos x$, for $0 \leq x \leq 360^\circ$	h $3 \tan x = 2 \sin x$, for $0 \leq x \leq 360^\circ$
- Solve these equations, giving all values of x to the nearest degree in the interval $0 \leq x \leq 360^\circ$.

a $2 \tan^2 x + \sec x - 1 = 0$	b $\operatorname{cosec}^2 x + \cot x - 2 = 0$
---------------------------------	---
- Find, in terms of π , the solutions of the equation $\tan^4 x - 4 \tan^2 x + 3 = 0$, for $0 \leq x \leq \pi$.
- Evaluate the following in degrees.

a $\sin^{-1} \frac{1}{\sqrt{2}}$	b $\cos^{-1} 1$	c $\tan^{-1} \sqrt{3}$
----------------------------------	-----------------	------------------------
- Evaluate the following in radians, in terms of π .

a $\cos^{-1} \frac{\sqrt{3}}{2}$	b $\tan^{-1} \frac{1}{\sqrt{3}}$	c $\sin^{-1} \frac{1}{2}$
----------------------------------	----------------------------------	---------------------------

- 6 Prove the following identities.
- $\operatorname{cosec} \theta \tan \theta \equiv \sec \theta$
 - $\operatorname{cosec} \theta - \sin \theta \equiv \cot \theta \cos \theta$
 - $(\sin \theta + \operatorname{cosec} \theta)^2 \equiv \sin^2 \theta + \cot^2 \theta + 3$
 - $\sec^2 \theta (\cot^2 \theta - \cos^2 \theta) \equiv \cot^2 \theta$
- 7 Solve the following equations in the range $0^\circ \leq \theta \leq 360^\circ$.
- $6 \sec^2 \theta = 5(\tan \theta + 1)$
 - $3 \cot \theta = 2 \sin \theta$
 - $\cot \theta + 3 \cos \theta = 0$
- 8 If $\cos A = \frac{4}{5}$, $\cos B = \frac{5}{13}$ and A and B are acute angles, find the value of $\cos(A + B)$.
- 9 The angle θ is obtuse and $\sin \theta = \frac{4}{5}$.
- Find the value of $\cos \theta$.
 - Find the value of $\sin 2\theta$ and $\cos 2\theta$, giving your answers as fractions in their lowest terms.
- 10 Prove the following identities.
- $\frac{\sin 2\theta}{1 + \cos 2\theta} \equiv \tan \theta$
 - $\frac{1 - \cos 2\theta}{1 + \cos 2\theta} \equiv \tan^2 \theta$
 - $\frac{\sin 2\theta + \sin \theta}{1 + \cos 2\theta + \cos \theta} \equiv \tan \theta$
 - $\frac{1 - \cos 2\theta + \sin 2\theta}{1 + \cos 2\theta + \sin 2\theta} = \tan \theta$
- 11 Solve the following equations in the range $0^\circ \leq x \leq 360^\circ$.
- $\cos 2\theta + \cos \theta = 0$
 - $3 \sin 2\theta = \sin \theta$
 - $2 \cos 2\theta = 3 \sin \theta$
 - $\sin 2\theta = \cos \theta$
 - $\cos 2\theta = \cos \theta$
 - $2 \sin 2\theta = 3 \tan \theta$
- 12 If θ is an acute angle such that $\cos \theta = \frac{3}{5}$ then find the exact values of:
- $\cos 2\theta$
 - $\sin 2\theta$
 - $\tan 2\theta$
- 13 a Find the exact values of the following (using surds):
- $\sin 15^\circ$
 - $\tan 75^\circ$
 - $\cos 105^\circ$
- b Hence show that $\sin 15^\circ + \cos 105^\circ = 0$.
- 14 If θ is an acute angle such that $\cos \theta = \frac{5}{13}$ then find the exact values of:
- $\cos 2\theta$
 - $\sin\left(\frac{\theta}{2}\right)$
 - $\tan 2\theta$
- 15 If θ is an acute angle such that $\tan \theta = \frac{3}{4}$ then find the exact value of $\tan\left(\frac{\theta}{2}\right)$:
- 16 Solve the equations in the range $0^\circ \leq \theta \leq 360^\circ$.
- $\cos 2\theta = \sin \theta$
 - $3 \tan \theta = \tan 2\theta$
 - $\cos 2\theta + \cos \theta + 1 = 0$
 - $2 \sin \theta = \tan \frac{1}{2} \theta$
 - $\tan \theta = 3 \tan \frac{1}{2} \theta$

17 Express $\sin 3\theta$ in terms of $\sin \theta$.

18 a Express $\cos x - \sin x$ in the form $R \cos(x + \alpha)$, $R > 0$, $0 < \alpha < \frac{\pi}{2}$. State the value of R in surd form and find the exact value of α in radians.

b Hence find the exact solutions in the interval $0 \leq x \leq 2\pi$ of the equation $\cos x - \sin x = 1$.

c State the maximum value of $\cos x - \sin x$.

19 a Express $\sin \theta^\circ + \sqrt{2} \cos \theta^\circ$ in the form $R \sin(\theta + \alpha)^\circ$, where $R > 0$ and $0 < \alpha < 90$.

b Solve the equation $\sin \theta + \sqrt{2} \cos \theta = 1$, for $0^\circ < \theta < 360^\circ$.

20 a Express $f(\theta) = \cos \theta - \sqrt{3} \sin \theta$ in the form $R \cos(\theta + \alpha)$, where $R > 0$ and $0 < \alpha < 90^\circ$.

b Hence find the least value of $\frac{1}{[f(\theta)]^2}$.

21 a Express $3 \cos \theta + 4 \sin \theta$ in the form $R \sin(\theta + \alpha)$, where $R > 0$ and $0 < \alpha < 90^\circ$. Give the value of α correct to one decimal place.

b Solve the equation $3 \cos \theta + 4 \sin \theta = 2$, for $0 < \theta < 360^\circ$, giving each solution to the nearest degree.

EXAMINATION EXERCISE 2

1 Find the solutions of

$$6 \tan \theta - \sec^2 \theta = 7$$

in the interval $0 \leq \theta < 2\pi$, giving each answer in radians to one decimal place.
[AQA]

2 i Express $\sec^2 \theta + \operatorname{cosec}^2 \theta$ in terms of $\sin \theta$ and $\cos \theta$, giving your answer as a single fraction as simply as possible.

ii Hence prove that

$$\sec^2 \theta + \operatorname{cosec}^2 \theta = 4 \operatorname{cosec}^2 2\theta.$$

iii Find the values of θ , for $0 < \theta < \frac{1}{2}\pi$, such that

$$\sec^2 \theta + \operatorname{cosec}^2 \theta = 10.$$

[OCR]

3 i Prove, by counter-example, that the statement

$$\sec(A + B) \equiv \sec A + \sec B, \text{ for all } A \text{ and } B$$

is false

ii Prove that

$$\tan \theta + \cot \theta \equiv 2 \operatorname{cosec} 2\theta, \theta \neq \frac{n\pi}{2}, n \in \mathbb{Z}.$$

[EDEXCEL]

4 a Prove that

$$\frac{2 \tan x}{1 + \tan^2 x} \equiv \sin 2x.$$

b Hence or otherwise find the exact value of $\tan 15^\circ$ in the form $a + b\sqrt{3}$, where a and b are integers. [AQA]

5 a Prove that

$$\frac{1 - \cos 2\theta}{\sin 2\theta} \equiv \tan \theta, \quad \theta \neq \frac{n\pi}{2}, \quad n \in \mathbb{Z}.$$

b Solve, giving exact answers in terms of π ,

$$2(1 - \cos 2\theta) = \tan \theta, \quad 0 < \theta < \pi. \quad [\text{EDEXCEL}]$$

6 a Show that $\frac{\cot^2 \theta}{1 + \cot^2 \theta} \equiv \cos^2 \theta$.

b Hence solve $\frac{\cot^2 \theta}{1 + \cot^2 \theta} = 2 \sin 2\theta$ for $0^\circ \leq \theta \leq 360^\circ$. [AQA]

7 Without using a calculator, find the exact value of

$$(\sin 22\frac{1}{2}^\circ + \cos 22\frac{1}{2}^\circ)^2. \quad [\text{OCR}]$$

8 a Express $6 \sin^2 \theta$ in the form $a + b \cos 2\theta$, where a and b are constants.

b Find the exact value of $\int_0^{\frac{\pi}{2}} 6 \sin^2 \theta \, d\theta$.

c Solve the equation

$$3 - 3 \cos 2\theta = 2 \operatorname{cosec} \theta$$

giving all solutions in radians in the interval $0 < \theta < 2\pi$. [AQA]

9 i Given that $\sin(\theta + 45^\circ) = 2 \sin \theta$, show that

$$\tan \theta = \frac{1}{2\sqrt{2} - 1}.$$

ii Hence solve the equation

$$\sin(\theta + 45^\circ) = 2 \sin \theta,$$

for values of θ between 0° and 360° , giving your answers correct to the nearest degree. [OCR]

10 a Show that $\tan(45^\circ + \theta) = \frac{1 + \tan \theta}{1 - \tan \theta}$.

b Hence obtain the exact value of $\tan 105^\circ$ in the form $a + b\sqrt{3}$, where a and b are integers to be found. [AQA]

11 i Write down the formula for $\tan 2x$ in terms of $\tan x$.

ii By letting $\tan x = t$, show that the equation

$$4 \tan 2x + 3 \cot x \sec^2 x = 0$$

becomes

$$3t^4 - 8t^2 - 3 = 0.$$

iii Find all the solutions of the equation

$$4 \tan 2x + 3 \cot x \sec^2 x = 0$$

which lie in the range $0 \leq x \leq 2\pi$.

[OCR]

12 a i By expanding $\cos(\frac{1}{4}\pi + \frac{1}{6}\pi)$, express $\cos \frac{5}{12}\pi$ in surd form.

ii Express $\tan(\frac{1}{4}\pi + x)$ in terms of $\tan x$.

iii Hence or otherwise show that

$$\int_0^{\frac{1}{6}\pi} \frac{1 + \tan x}{1 - \tan x} dx = \ln\left(\frac{2}{\sqrt{3} - 1}\right).$$

[OCR]

13 a Express $\sin x + \sqrt{3} \cos x$ in the form $R \sin(x + \alpha)$, where $R > 0$ and $0 < \alpha < 90^\circ$.

b Show that the equation $\sec x + \sqrt{3} \operatorname{cosec} x = 4$ can be written in the form

$$\sin x + \sqrt{3} \cos x = 2 \sin 2x.$$

c Deduce from parts a and b that $\sec x + \sqrt{3} \operatorname{cosec} x = 4$ can be written in the form

$$\sin 2x - \sin(x + 60^\circ) = 0.$$

[EDEXCEL]

14 a Given that $\tan x \neq 1$, show that

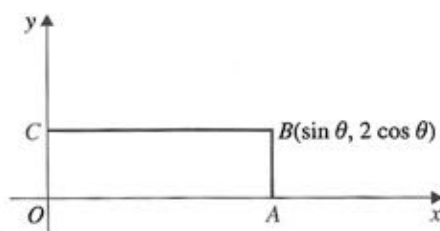
$$\frac{\cos 2x}{\cos x - \sin x} = \cos x + \sin x.$$

b By expressing $\cos x + \sin x$ in the form $R \sin(x + a)$, solve, for $0^\circ \leq x \leq 360^\circ$,

$$\frac{\cos 2x}{\cos x - \sin x} = \frac{1}{2}.$$

[AQA]

15 The diagram shows a rectangle $OABC$ in which B has coordinates $(\sin \theta, 2 \cos \theta)$, where $0 \leq \theta \leq \frac{\pi}{2}$. The perimeter of the rectangle is of length L .



a i Write down the length L in terms of θ .

ii Hence obtain an expression for L in the

form $R \sin(\theta + \alpha)$, where $R > 0$ and $0 \leq \alpha \leq \frac{\pi}{2}$.

Give your answer for α to three decimal places.

b Given that θ varies between 0 and $\frac{\pi}{2}$:

i write down the maximum value of L ;

ii find the value of θ , to two decimal places, for which L is maximum.

[AQA]

- 16 i In the diagram, $ABCD$ represents a rectangular table with sides 3.5 m and 1.5 m. It has been turned so that it wedges in a passage of width 2.5 m. Given that θ° is the acute angle between the longer side and the passage, as shown in the diagram, show clearly why

$$7 \sin \theta^\circ + 3 \cos \theta^\circ = 5.$$

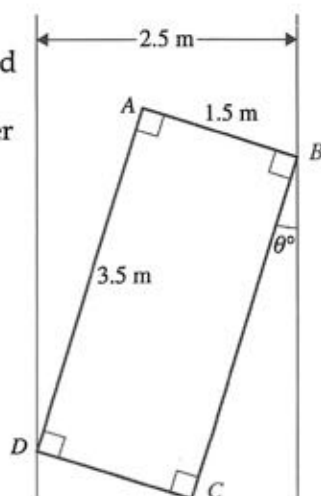
- ii Express $7 \sin \theta^\circ + 3 \cos \theta^\circ$ in the form

$$R \sin(\theta + \alpha)^\circ,$$

where $R > 0$ and $0 < \alpha < 90$.

- iii Find θ .

[OCR]



- 17 a Express $2 \cos \theta + 5 \sin \theta$ in the form $R \cos(\theta - \alpha)$, where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$. Give the values of R and α to 3 significant figures.
- b Find the maximum and minimum values of $2 \cos \theta + 5 \sin \theta$ and the smallest possible value of θ for which the maximum occurs.

The temperature $T^\circ\text{C}$, of an unheated building is modelled using the equation

$$T = 15 + 2 \cos\left(\frac{\pi t}{12}\right) + 5 \sin\left(\frac{\pi t}{12}\right), \quad 0 \leq t < 24,$$

where t hours is the number of hours after 1200.

- c Calculate the maximum temperature predicted by this model and the value of t when this maximum occurs.
- d Calculate, to the nearest half hour, the times when the temperature is predicted to be 12°C .

[EDEXCEL]

- 18 The diagram shows the graphs of $y = \cos^{-1} x$ and $y = \tan^{-1} x$, for $-1 \leq x \leq 1$ in each case. The graphs intersect at the point with coordinates (p, q) .

- i State the exact values of y for $x = -1, 0$ and 1 on each graph.
- ii Write down, in terms of p and q , the coordinates of the corresponding point of intersection of the graphs of $y = \cos x$ and $y = \tan x$, and hence show that $\cos^2 q = \sin q$.
- iii Deduce that $p^4 + p^2 - 1 = 0$.
- iv Hence find in exact form the solution of the equation

$$\cos^{-1} x = \tan^{-1} x.$$

[OCR]

